A note on a representation of a Poisson(1/2) random variable as a sum of Bernoulli products

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May 11, 2020

1 Introduction

This main goal of this note is to attempt to answer a question posed by Persi Diaconis in a lecture for his "Topics in Combinatorics" course. The question is motivated by the following observation.

Proposition 1. *Let $X_1, X_2, \ldots$ be independent Bernoulli random variables, where each X_i is 1 with probability $1/i$. Then $\sum_{i=1}^{\infty} X_i X_{i+1} \overset{d}{=} \text{Poisson}(1)$.*

We will review a proof of this result in Section 2, which will reveal the key observation — the sum can be related the number of fixed points in a random permutation on n elements, which has a limiting Poisson(1) distribution as $n \to \infty$.

It is also known that the number of transpositions in a random permutation has a limiting Poisson(1/2) distribution. We should thus expect to find a similar representation of the Poisson(1/2). In this note, we show that, in fact, the following is true.

Proposition 2. *With $X_1, X_2, \ldots$ as in Proposition 1, and denoting $\overline{X_i} = 1 - X_i$, we have $\sum_{i=1}^{\infty} X_i \overline{X_{i+1}} X_{i+2} \overset{d}{=} \text{Poisson}(1/2)$.*

Note that throughout, we will use X_i and $\overline{X_i}$ with the meanings above.

This result is certainly not new, being a special case of one for general Poisson($1/k$) distributions, which can be proved by the Feller coupling (cf. Najnudel and J. Pitman [2]).[1] In this note, we provide a different argument via random permutation matrices. We begin by reviewing a version of this argument for the Poisson(1) case, then generalize it to the Poisson(1/2), and conclude by proposing a similar approach to the general Poisson($1/k$) case.

[1] For a non-combinatorial proof, see Sethuraman and Sethuraman [3].

2 A random matrix argument for the Poisson(1) case

In this section, we review a proof of Proposition 1. We will begin by outlining the key intermediate results before going into their respective proofs.

As mentioned above, the proof relies on the fact that the number of fixed points in a random permutation has a limiting Poisson(1) distribution. This is in fact a special case of the following classical result.

Proposition 3. *Let $\sigma \in S_n$ be a uniformly random permutation of n elements, and let $a_i(\sigma)$ denote the number of cycles in σ of length k. The cycle counts $a_k(\sigma)$ have limiting* Poisson$(1/k)$ *limiting distributions as $n \to \infty$.*

This can be seen as a fact about the traces of random permutation matrices. Letting $\rho(\sigma)$ denote the usual matrix representation of σ, it is easy to see that the number of element which are in cycles of length dividing k is given by $\mathrm{Tr}(\rho(\sigma)^k)$. In particular, the above result says that for a random $\sigma \in S_n$, we have $\mathrm{Tr}(\rho(\sigma)) \xrightarrow{d} \mathrm{Poisson}(1)$ as $n \to \infty$.

Given the result above, to prove Proposition 1 it remains to show $\mathrm{Tr}(\rho(\sigma)) \xrightarrow{d} \sum_{i=1}^{\infty} X_i X_{i+1}$. This is a direct consequence of the following result.

Proposition 4. *For a uniformly random $\sigma \in S_n$, $\mathrm{Tr}(\rho(\sigma)) \stackrel{d}{=} X_n + \sum_{i=1}^{n} X_i X_{i+1}$.*

For Proposition 3, we follow an argument of Diaconis and Shashahani [1]. We will strive to be scrupulous and fill in details, beginning with the following.

Lemma 5. *Let $\alpha_1, \ldots, \alpha_n$ be non-negative integers with $\sum_{k=1}^{n} \alpha_k = n$. The number of permutations $\sigma \in S_n$ such that $a_k(\sigma) = \alpha_k$ for all $k \in [n]$ is*

$$n! \prod_{k=1}^{n} \frac{1}{\alpha_k! k^{\alpha_k}}.$$

Proof. This (among other ways) can be seen by an elementary counting argument. There are $n!$ permutations in S_n, each of which can be represented as a tableau. These tableaux are unique only up to cyclic shifts within rows and switching rows of the same size. Each of the α_k rows of size k can be shifted k times, and there are $\alpha_k!$ permutations of such rows, giving the above result. □

Proof of Proposition 3. Let us denote the *cycle generating functions*:

$$C_n(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{k=1}^{n} x_k^{a_k(\sigma)}$$

$$C(t)(x_1, x_2, \ldots) = \sum_{n=0}^{\infty} t^n C_n(x_1, \ldots, x_n).$$

Note that we have

$$
\begin{aligned}
C_n(x_1, \ldots, x_n) &= \frac{1}{n!} \sum_{\{\alpha_k\}} |\{\sigma \in S_n : a_k(\sigma) = \alpha_k \ \forall k\}| \left(\prod_{k=1}^{n} x_k^{\alpha_k} \right) \\
&= \sum_{\{\alpha_k : \sum_k k\alpha_k = n\}} \prod_{k=1}^{n} \frac{x_k^{\alpha_k}}{\alpha_k! k^{\alpha_k}},
\end{aligned}
$$

the second equality holding by our previous result. Thus,

$$
\begin{aligned}
C(t)(x_1, x_2, \ldots) &= \sum_{n=0}^{\infty} t^n \sum_{\{\alpha_k : \sum_k k\alpha_k = n\}} \prod_{k=1}^{n} \frac{x_k^{\alpha_k}}{\alpha_k! k^{\alpha_k}} \\
&= \prod_{k=1}^{\infty} \sum_{\alpha_k=0}^{\infty} \frac{(x_k t^k / k)^{\alpha_k}}{\alpha_k!} \\
&= \prod_{k=1}^{\infty} e^{x_k t^k / k}.
\end{aligned}
$$

We will use our two different expressions for $C(t)(x_1, x_2 \ldots)$ to derive the result. In particular, we will show that for a uniformly random $\sigma \in S_n$ and $Y \sim \text{Poisson}(1/k)$, for any cycle length k and $j \leq n/k$, we have $\mathbb{E}[a_k(\sigma)_{(j)}] = \mathbb{E}[Y_{(j)}]$. That is, the j-th falling factorial moments are equal. It follows that the first n moments are in fact equal.

To see this, write x for x_k and fix all other $x_i = 1$, and note that $\mathbb{E}[x^{a_k(\sigma)}] = C_n(x)$. It follows that

$$
\mathbb{E}[a_k(\sigma)_{(j)}] = \frac{d^j}{dx^j} C_{n,k}(x) \bigg|_{x=1}
$$

and furthermore

$$
\frac{d^j}{dx^j} C(t)(x) \bigg|_{x=1} = \sum_{n=0}^{\infty} t^n \mathbb{E}[a_k(\sigma)_{(j)}].
$$

On the other hand, we have

$$
C(t)(x) = \frac{e^{\frac{t^k}{k}(x-1)}}{1-t}.
$$

Differentiating, we obtain

$$
\sum_{n=0}^{\infty} t^n \mathbb{E}[a_k(\sigma)_{(j)}] = \frac{(t^k/k)^j}{1-t} = \left(\frac{1}{k} \right)^j \sum_{n=kj}^{\infty} t^n,
$$

finishing the proof. $\square$

Our proof of Proposition 4 relies on the following representation of a uniformly random permutation.

Lemma 6. *For $i \in \{2, \ldots, n\}$ and $j \in [n-1]$, let $X_i^{(j)} \sim Ber(1/i)$ be independent and let $T_i^{(j)}$ be the $n \times n$ matrix representing the transposition $(i-1, i)$ if $X_i^{(j)} = 0$, and the identity if $X_i^{(j)} = 1$. Then*

$$(T_2^{(1)} \cdots T_n^{(1)})(T_2^{(2)} \cdots T_{n-1}^{(2)}) \cdots (T_2^{(n-1)})$$

is the matrix representation of a permutation uniformly random on S_n.

Proof. Observe that $T_2^{(j)} \cdots T_i^{(j)}$ sends index i to i' (where $i' \leq i$) exactly when all transpositions $T_i^{(j)}, \ldots, T_{i'+1}^{(j)}$ are non-trivial and $T_{i'}^{(j)}$ is the identity. This is true exactly when $\overline{X_i} \cdots \overline{X_{i'+1}} X_{i'} = 1$, which occurs with probability $\frac{i-1}{i}\frac{i-2}{i-1} \cdots \frac{i'}{i'+1}\frac{1}{i'} = \frac{1}{i}$. That is, i is uniformly random in $[i]$. The result holds by induction. □

We proceed now by showing that the above product representation of a uniform permutation has trace equal in distribution to $\mathrm{Tr}(T_n \cdots T_2)$, for T_i as above. (We omit the superscripts (j) where they may be fixed without affecting the end result.) Proposition 4 then follows from the observation that $\mathrm{Tr}(\rho(T_n \cdots T_2)) = X_n + \sum_{i=1}^n X_i X_{i+1}$.

Proof of Proposition 4. For $T_i^{(j)}$ as in Lemma 6, we show

$$\mathrm{Tr}\left((T_2^{(1)} \cdots T_n^{(1)})(T_2^{(2)} \cdots T_{n-1}^{(2)}) \cdots (T_2^{(n-1)})\right) \stackrel{d}{=} \mathrm{Tr}(T_n \cdots T_2).$$

We show this by iterating a procedure of "cycling" the transpositions and "eliminating" (i.e. replacing by an expression equal in distribution), repeated $n-1$ times.

Note that the right hand side above, by the cyclic property of trace, is equal to

$$\mathrm{Tr}\left((T_n^{(1)})(T_2^{(2)} \cdots T_{n-1}^{(2)}) \cdots (T_2^{(n-1)}) T_2^{(1)} \cdots T_{n-1}^{(1)}\right).$$

Noting that, by Lemma 6, $(T_2^{(2)} \cdots T_{n-1}^{(2)}) \cdots (T_2^{(n-1)})$ is the representation of a uniformly random element of S_{n-1}, the above is equal in distribution to

$$\mathrm{Tr}\left((T_n^{(1)})(T_2^{(2)} \cdots T_{n-1}^{(2)}) \cdots (T_2^{(n-1)})\right).$$

The above is what we call the "elimination" step. Now, since the transposition matrices $T_2^{(2)} \cdots T_{n-2}^{(2)}$ do not interact with indices $n-1$ and n, they commute with $T_n^{(1)}$, and the above is thus equal to

$$\mathrm{Tr}\left((T_n^{(1)})(T_{n-1}^{(2)})(T_2^{(3)} \cdots T_{n-2}^{(3)}) \cdots (T_2^{(n-1)}) T_2^{(2)} \cdots T_{n-2}^{(2)}\right).$$

The "elimination" argument applies once more, showing the above is equal in distribution to

$$\mathrm{Tr}\left((T_n^{(1)})(T_{n-1}^{(2)})(T_2^{(3)} \cdots T_{n-2}^{(3)}) \cdots (T_2^{(n-1)})\right).$$

It is evident that this procedure may be repeated until we obtain the expression

$$\mathrm{Tr}(T_n^{(1)} \cdots T_2^{(n-1)}).$$

□

3 Generalizing to Poisson(1/2)

We now generalize the argument made above to prove Proposition 2. The obvious step to make is to write an expression for $a_2(\sigma)$ in terms of $\rho(\sigma)$, and prove it converges in distribution to $\sum_{i=1}^{\infty} X_i\overline{X_{i+1}}X_{i+2}$. But in fact, it is easy to see that $a_2(\sigma) = \frac{1}{2}(\mathrm{Tr}(\rho(\sigma)^2) - \mathrm{Tr}(\rho(\sigma)))$. (That is, count the number of elements in cycles of lengths dividing 2, subtract the number of fixed points, and divide the result by two.) Proposition 2 thus holds given the following result.

Proposition 7. *For a uniformly random $\sigma \in S_n$,*

$$\mathrm{Tr}(\rho(\sigma)^2) \stackrel{d}{=} X_n + 2X_{n-1}\overline{X_n} + \sum_{i=1}^{n-2} X_iX_{i+1} + 2X_i\overline{X_{i+1}}X_{i+2}.$$

Proof. As before, write $\rho(\sigma)$ as a product of transposition matrices

$$\rho(\sigma) = (T_2^{(1)}\cdots T_n^{(1)})(T_2^{(2)}\cdots T_{n-1}^{(2)})\cdots(T_2^{(n-1)}).$$

We claim that $\mathrm{Tr}(\rho(\sigma)^2) \stackrel{d}{=} \mathrm{Tr}\left((T_n\cdots T_2)^2\right)$. In fact, essentially the same argument as in the proof of Proposition 4 applies; the only difference is that both copies of transpositions with the same superscript index are "cycled" through the product at once and (crucially!) both copies are then "eliminated" together. As an example, we work through the case $n = 4$:

$$\begin{aligned}
&\mathrm{Tr}\left((T_2^{(1)}T_3^{(1)}T_4^{(1)})(T_2^{(2)}T_3^{(2)})(T_2^{(3)})(T_2^{(1)}T_3^{(1)}T_4^{(1)})(T_2^{(2)}T_3^{(2)})(T_2^{(3)})\right)\\
&= \mathrm{Tr}\left((T_4^{(1)})(T_2^{(2)}T_3^{(2)})(T_2^{(3)})(T_2^{(1)}T_3^{(1)}T_4^{(1)})(T_2^{(2)}T_3^{(2)})(T_2^{(3)})(T_2^{(1)}T_3^{(1)})\right)\\
&\stackrel{d}{=} \mathrm{Tr}\left((T_4^{(1)})(T_2^{(2)}T_3^{(2)})(T_2^{(3)})(T_4^{(1)})(T_2^{(2)}T_3^{(2)})(T_2^{(3)})\right)\\
&= \mathrm{Tr}\left((T_4^{(1)})(T_3^{(2)})(T_2^{(3)})(T_2^{(2)})(T_4^{(1)})(T_3^{(2)})(T_2^{(3)})(T_2^{(2)})\right)\\
&\stackrel{d}{=} \mathrm{Tr}\left((T_4^{(1)})(T_3^{(2)})(T_2^{(3)})(T_4^{(1)})(T_3^{(2)})(T_2^{(3)})\right).
\end{aligned}$$

Letting again each transposition T_i be the identity when $X_i = 1$, one may verify that

$$\mathrm{Tr}\left((T_n\cdots T_2)^2\right) = X_n^2 + 2X_{n-1}\overline{X_n}^{\,2} + \sum_{i=1}^{n-2} X_i^2X_{i+1}^2 + 2X_i\overline{X_{i+1}}^{\,2}X_{i+2}.$$

Noting that $X_i \in \{0,1\}$ and that we may thus ignore exponents finishes the proof. $\square$

4 Concluding remarks

It should be possible to generalize this proof to a representation of any Poisson$(1/k)$ variable, e.g. by observing the recurrence $a_k(\sigma) = \frac{1}{k}(\mathrm{Tr}(\rho(\sigma)^k) - \sum_{j|k} ja_j(\sigma))$, where the sum is over divisors $j \neq k$ of k. As a step in this direction, one should be able to directly generalize the "cycling" argument to show $\mathrm{Tr}(\rho(\sigma)^k) \overset{d}{=} \mathrm{Tr}\left((T_n \cdots T_2)^k\right)$. Ideally, however, a less mechanistic argument than the one by "cycling-and-elimination" may be made (perhaps via character theory).

References

[1] Persi Diaconis and Mehrdad Shahshahani. "On the Eigenvalues of Random Matrices". In: *Journal of Applied Probability* 31 (1994), pp. 49–62. ISSN: 00219002. URL: `http://www.jstor.org/stable/3214948`.

[2] Joseph Najnudel and Jim Pitman. *Feller coupling of cycles and Poisson spacings*. 2019. arXiv: `1907.09587 [math.PR]`.

[3] Jayaram Sethuraman and Sunder Sethuraman. "On counts of Bernoulli strings and connections to rank orders and random permutations". In: *A Festschrift for Herman Rubin*. Ed. by Anirban DasGupta. Vol. Volume 45. Lecture Notes–Monograph Series. Beachwood, Ohio, USA: Institute of Mathematical Statistics, 2004, pp. 140–152. DOI: `10.1214/lnms/1196285386`. URL: `https://doi.org/10.1214/lnms/1196285386`.